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Nonlinear solutions of modified plane Couette flow in the presence of a transverse magnetic field

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A nonlinear analysis is performed numerically for the motion of an electrically conducting fluid between parallel plates in relative motion when a transverse magnetic field is applied. It is found that steady three-dimensional finite-amplitude solutions exist even when the linear analysis predicts an infinite critical Reynolds number.

1. Introduction

Thirty years ago Kakutani (1964) investigated the linear stability of a hydromagnetic plane Couette flow modified by the presence of a transverse magnetic field with the following results. As in the case of purely hydrodynamic plane Couette flows the critical Reynolds number is infinite for any wavenumber as long as the Hartmann number H, which measures the strength of the applied magnetic field, is moderate ($0 \leqslant H \leqslant 7.82$). As the Hartmann number is increased from 7.82 the critical Reynolds number decreases sharply from the infinite value, indicating the destabilizing effect of the magnetic field. The minimum value of the critical Reynolds number occurs when H is 10.8. Further increase of H results in the increase of the critical Reynolds number to its asymptotic value so that the magnetic field stabilizes the flow for $H > 10.8$. Growing disturbances in the range of the Hartmann number greater than 7.82 are all characterized by oscillatory instabilities. These results are obtained by applying the Squire theorem so that only two-dimensional disturbances are considered.

It should be emphasized that the understanding of the interaction between shear flows and magnetic fields is one of the decisive factors in designing heat transfer machines such as magnetohydrodynamic(MHD) power generators and nuclear fusion devices (Girshick & Kruger (1986) and Molokov (1993) and references therein). Also, it is widely believed that the magnetic field of the Earth is strongly influenced by the presence of hydrodynamic shear motions in the liquid core. Yet, in spite of the recent progress in geophysical and astrophysical MHD studies (see, for instance, Proctor & Gilbert 1994), nonlinear analyses of a simple system such as was considered by Kakutani (1964) have not been developed so far. In the present paper three-dimensional finite-amplitude solutions discovered recently in the case of purely hydrodynamic plane Couette flows by Nagata (1990) are continued to the magnetohydrodynamic case.

2. Formulation of the problem

We consider an electrically conducting fluid of electric conductivity σ and magnetic permeability μ between two parallel plates separated by the distance L. The bottom

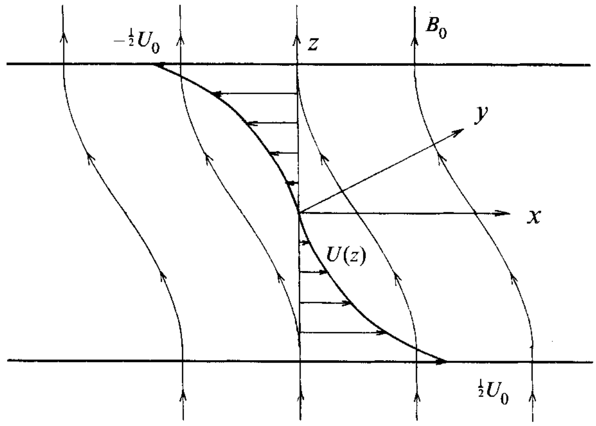

FIGURE 1. Physical configuration.

plate moves along in its own plane, indicated by the x-direction in figure 1, with a constant speed $\frac{1}{2}U_0$ whereas the top plate moves in the negative x-direction with speed $-\frac{1}{2}U_0$. Both plates are electrically insulated. Owing to the viscosity ν of the fluid a shear motion is induced across the fluid layer and the profile of the motion is modified by an applied transverse magnetic field B_0 through the Lorentz force. We assume that the fluid is incompressible with a constant density ρ. Then, the motion is governed by the equation of continuity,

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.1}$$

the conservation of momentum,

$$\frac{\partial \boldsymbol{u}}{\partial t} + R\boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nabla^2 \boldsymbol{u} + RA^2(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{2.2}$$

the absence of magnetic monopoles,

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.3}$$

and the induction equation,

$$P_m \left\{ \frac{\partial \boldsymbol{B}}{\partial t} - R\nabla \times (\boldsymbol{u} \times \boldsymbol{B}) \right\} = \nabla^2 \boldsymbol{B}, \tag{2.4}$$

where the velocity $\boldsymbol{u}$ and the magnetic field $\boldsymbol{B}$ are normalized by U_0 and B_0, respectively. Time t and the position vector $\boldsymbol{r} = (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}})$ with $\hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{k}}$ being the unit direction vectors are already nondimensionalized in the equations above by using the time scale for the viscous dissipation L^2/ν and the length scale L. The nondimensional parameters controlling the system are the Reynolds number,

$$R = \frac{U_0 L}{\nu}, \tag{2.5}$$

the Alfvén number,

$$A = \frac{1}{(\rho\mu)^{1/2}} \frac{B_0}{U_0}, \tag{2.6}$$

and the magnetic Prandtl number,

$$P_m = \mu\sigma\nu. \tag{2.7}$$

In order to simplify the problem we assume that the magnetic Prandtl number P_m is small but the Alfvén number A is large enough for the Hartmann number H defined by

$$H = P_m^{1/2} R A \tag{2.8}$$

to retain its finite values (see Chapter 5 of Roberts 1967). The limit of small Prandtl number is appropriate for liquid metals used in laboratory experiments or in MHD power generators. The limit is also applicable in the liquid core of the Earth and magnetic stars. Finite Hartmann numbers are characteristic flows in these situations.

The no-slip condition for the velocity and the insulating condition for the magnetic field are given on the plates at $z = \pm\frac{1}{2}$:

$$\boldsymbol{u} = \mp\tfrac{1}{2}\hat{\boldsymbol{i}}, \tag{2.9}$$

$$\hat{\boldsymbol{k}} \cdot \nabla \times \boldsymbol{B} = 0. \tag{2.10}$$

In addition, the magnetic field of the fluid must be matched to the potential field in the insulating exterior. Expansions of $\boldsymbol{u}$, $\boldsymbol{B}$, and the pressure p in powers of $P_m \ll 1$, i.e.

$$\boldsymbol{u} = \boldsymbol{u}^{(0)} + P_m \boldsymbol{u}^{(1)} + P_m^2 \boldsymbol{u}^{(2)} + \cdots, \tag{2.11}$$

$$\boldsymbol{B} = \boldsymbol{b}^{(0)} + P_m \boldsymbol{b}^{(1)} + P_m^2 \boldsymbol{b}^{(2)} + \cdots, \tag{2.12}$$

$$p = p^{(0)} + P_m p^{(1)} + P_m^2 p^{(2)} + \cdots, \tag{2.13}$$

yield the following equations at the lowest order:

$$\nabla \cdot \boldsymbol{u}^{(0)} = 0, \tag{2.14}$$

$$(\nabla \times \boldsymbol{b}^{(0)}) \times \boldsymbol{b}^{(0)} = 0 \tag{2.15}$$

$$\nabla \cdot \boldsymbol{b}^{(0)} = 0, \tag{2.16}$$

$$\nabla^2 \boldsymbol{b}^{(0)} = 0, \tag{2.17}$$

with the boundary conditions

$$\boldsymbol{u}^{(0)} = \mp\tfrac{1}{2}\hat{\boldsymbol{i}}, \tag{2.18}$$

$$\hat{\boldsymbol{k}} \cdot \nabla \times \boldsymbol{b}^{(0)} = 0 \tag{2.19}$$

at $z = \pm\frac{1}{2}$. From (2.17) and (2.19) $\boldsymbol{b}^{(0)}$ is immediately solved:

$$\boldsymbol{b}^{(0)} \equiv \hat{\boldsymbol{k}}. \tag{2.20}$$

The magnetic field in the exterior must be constant if it is independent of the coordinates, x and y.

Substitution of (2.20) into (2.2), the solenoidal condition for $\boldsymbol{b}^{(1)}$ and the inclusion of the induction equation at the next order close the system. The final equations to be solved are

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.21}$$

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nabla^2 \boldsymbol{u} + H^2 (\nabla \times \boldsymbol{b}) \times \hat{\boldsymbol{k}}, \tag{2.22}$$

$$\nabla \cdot \boldsymbol{b} = 0, \tag{2.23}$$

$$\nabla^2 \boldsymbol{b} = -(\hat{\boldsymbol{k}} \cdot \nabla) \boldsymbol{u}, \tag{2.24}$$

subject to the boundary conditions

$$u = \mp\tfrac{1}{2}R\hat{\imath}, \tag{2.25}$$

$$\hat{k}\cdot\nabla\times b = 0 \tag{2.26}$$

at $z = \pm\frac{1}{2}$, where the superscripts of $u^{(0)}$, $b^{(1)}$ and $p^{(0)}$ have been dropped after $u^{(0)}$ and $p^{(0)}$ are rescaled by the Reynolds number, i.e. $u = Ru^{(0)}$ and $p = Rp^{(0)}$. The system accommodates the basic laminar flow solution

$$u = U(z)\hat{\imath}, \qquad b = B(z)\hat{\imath}, \tag{2.27}$$

with

$$U(z) = -\frac{R\sinh Hz}{2\sinh H/2}, \tag{2.28}$$

and

$$B(z) = -\frac{R\coth H/2}{2H}\left(1 - \frac{\cosh Hz}{\cosh H/2}\right). \tag{2.29}$$

Note that (2.29) satisfies the condition $B(\pm\frac{1}{2}) = 0$, i.e. the tangential component of the magnetic field is continuous across the insulating boundaries.

We are interested in the situation where the fields, u and b, deviate from this basic state. After substituting perturbed fields with $\check{u}$ and $\check{b}$ being fluctuations, we operate with $\hat{k}\cdot\nabla\times\nabla\times$, $\hat{k}\cdot\nabla\times$ on (2.22) and $\hat{k}\cdot\nabla\times$, $\hat{k}\cdot$ on (2.24). Then, we obtain

$$(\hat{U}\partial_x - \nabla^2 + \partial_t)\nabla^2\Delta_2\phi - \hat{U}''\partial_x\Delta_2\phi - H^2\partial_z\nabla^2\Delta_2 h + \hat{k}\cdot\nabla\times\nabla\times[\check{u}\cdot\nabla\check{u}] = 0, \tag{2.30}$$

$$(\hat{U}\partial_x - \nabla^2 + \partial_t)\Delta_2\psi - \hat{U}'\partial_y\Delta_2\phi - H^2\partial_z\Delta_2 g - \hat{k}\cdot\nabla\times[\check{u}\cdot\nabla\check{u}] = 0, \tag{2.31}$$

$$\nabla^2\Delta_2 h = -\partial_z\Delta_2\phi, \tag{2.32}$$

$$\nabla^2\Delta_2 g = -\partial_z\Delta_2\psi, \tag{2.33}$$

where ∂ is the partial differential operator with respect to its subscript, $\Delta_2 = \partial^2_{xx} + \partial^2_{yy}$ is the two-dimensional Laplacian operator, and the prime denotes differentiation with respect to z. (When the flow is time-independent, the prime denotes ordinary differentiation with respect to z.) The mean fields, $\hat{U}(t,z)$ and $\hat{B}(t,z)$, include modifications $\check{U}(t,z)$ and $\check{B}(t,z)$ due to the nonlinear nature of the system:

$$\hat{U}(t,z) = U(z) + \check{U}(t,z), \qquad \hat{B}(t,z) = B(z) + \check{B}(t,z). \tag{2.34}$$

The poloidal and the toroidal parts of the disturbed solenoidal fields $\check{u}$ and $\check{b}$ whose xy-averages are zero are denoted by ϕ, ψ and h, g, respectively, so that the total fields, u and b, now become

$$u = (U(z) + \check{U}(t,z))\hat{\imath} + \nabla\times\nabla\times\hat{k}\phi + \nabla\times\hat{k}\psi, \tag{2.35}$$

$$b = (B(z) + \check{B}(t,z))\hat{\imath} + \nabla\times\nabla\times\hat{k}h + \nabla\times\hat{k}g. \tag{2.36}$$

The equations for $\check{U}$ and $\check{B}$ are obtained by taking the xy-average of the x-component of (2.22) and (2.24):

$$-\partial_t\check{U} + \partial^2_{zz}\check{U} + H^2\partial_z\check{B} = -\partial_z\overline{\Delta_2\phi(\partial^2_{xz}\phi + \partial_y\psi)}, \tag{2.37}$$

$$\partial^2_{zz}\check{B} = -\partial_z\check{U}. \tag{2.38}$$

The boundary conditions at $z = \pm\frac{1}{2}$ are

$$\phi = \partial_z\phi = \psi = g = \check{U} = \check{B} = 0. \tag{2.39}$$

Since h can be eliminated by (2.30) and (2.31), there is no need to specify the boundary conditions for it as far as the detection of nonlinear solutions is concerned.

3. The symmetry

By using (2.32) and (2.33), those terms involving h and g in (2.30) and (2.31) are readily expressed in terms of ϕ and ψ with the result that the symmetry with respect to z is unaltered. To see this, compare these terms with, for example, the viscous term in each equation: the symmetry of h (or g) operated on by ∂_z an odd number of times is related to the symmetry of ϕ (or ψ) with the same operation an even number of times. The Laplacian operators do not change the symmetry.

By taking into account of the fact that $\check{U}$ is antisymmetric in z, we solve (2.38) with the boundary conditions for $\check{U}$ and $\check{B}$:

$$\check{B}(t,z) = -\int_{-1/2}^{z} \check{U}(t,z)\mathrm{d}z. \tag{3.1}$$

The above argument leads to the following three equations for ϕ, ψ and $\check{U}$:

$$(\hat{U}\partial_x - \nabla^2 + \partial_t)\nabla^2\Delta_2\phi - \hat{U}''\partial_x\Delta_2\phi + H^2\partial_{zz}^2\Delta^2\phi + \hat{\boldsymbol{k}}\cdot\nabla\times\nabla\times[\check{\boldsymbol{u}}\cdot\nabla\check{\boldsymbol{u}}] = 0, \tag{3.2}$$

$$(\hat{U}\partial_x - \nabla^2 + \partial_t)\Delta_2\psi - \hat{U}'\partial_y\Delta_2\phi + H^2\partial_{zz}^2\nabla^{-2}\Delta_2\psi - \hat{\boldsymbol{k}}\cdot\nabla\times[\check{\boldsymbol{u}}\cdot\nabla\check{\boldsymbol{u}}] = 0, \tag{3.3}$$

$$-\partial_t\check{U} + \partial_{zz}^2\check{U} - H^2\check{U} + \partial_z\overline{\Delta_2\phi(\partial_{xz}^2\phi + \partial_y\psi)} = 0, \tag{3.4}$$

with the boundary conditions

$$\phi = \partial_z\phi = \psi = \check{U} = 0 \tag{3.5}$$

at $z = \pm\frac{1}{2}$. The inverse Laplacian operator ∇^{-2} in (3.3) is given in the Appendix. Comparison of (3.2), (3.3) and (3.4) with the basic equations for three-dimensional nonlinear hydrodynamic plane Couette flows (see equations (1), (2) and (3) of Nagata 1990 with $\Omega = 0$) leads to the conclusion that the effects of the magnetic field, namely those additional terms with H^2 as a factor in the equations above, do not change the symmetry of the pure hydrodynamic case in the limit considered in the present paper.

As in the purely hydrodynamic case considered by Nagata (1990), only steady solutions are of interest here. First, he expanded ϕ, ψ and $\check{U}$ in each of the three spatial directions in terms of an appropriate set of orthogonal functions, i.e.

$$\phi(x,y,z) = \sum_{\ell=1}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} a_{\ell mn}\mathrm{e}^{\mathrm{i}m\alpha x}\mathrm{e}^{\mathrm{i}n\beta y}f_\ell(z), \tag{3.6}$$

$$\psi(x,y,z) = \sum_{\ell=1}^{\infty}\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} b_{\ell mn}\mathrm{e}^{\mathrm{i}m\alpha x}\mathrm{e}^{\mathrm{i}n\beta y}\sin\ell\pi(z+\tfrac{1}{2}), \tag{3.7}$$

$$\check{U}(z) = \sum_{k=1}^{\infty} c_k \sin 2k\pi(z+\tfrac{1}{2}), \tag{3.8}$$

where

$$f_\ell(z) = \begin{cases} \dfrac{\cosh \nu_{(\ell+1)/2} z}{\cosh \frac{1}{2}\nu_{(\ell+1)/2}} - \dfrac{\cos \nu_{(\ell+1)/2} z}{\cos \frac{1}{2}\nu_{(\ell+1)/2}} & (\ell \text{ odd}) \\ \dfrac{\sinh \mu_{\ell/2} z}{\sinh \frac{1}{2}\mu_{\ell/2}} - \dfrac{\sin \mu_{\ell/2} z}{\sin \frac{1}{2}\mu_{\ell/2}} & (\ell \text{ even}) \end{cases} \tag{3.9}$$

with ν_i and μ_i, $(i = 1, 2, \cdots)$ being the roots of

$$\tanh \tfrac{1}{2}\nu_i + \tan \tfrac{1}{2}\nu_i = 0, \tag{3.10}$$

$$\coth \tfrac{1}{2}\mu_i - \cot \tfrac{1}{2}\mu_i = 0. \tag{3.11}$$

$f_\ell(z)$ are known as the Chandrasekhar functions (Chandrasekhar 1961), which satisfy $f(\pm\frac{1}{2}) = f'(\pm\frac{1}{2}) = 0$. Next, he restricted his attention to the following set of interacting components to obtain finite-amplitude steady three-dimensional solutions in plane Couette flow:

$$\phi \propto \begin{cases} \cos m'\alpha x \cos n'\beta y\, f_{\ell'}(z) \\ \cos m''\alpha x \cos n''\beta y\, f_{\ell''}(z) \\ \cos m'\alpha x \sin n''\beta y\, f_{\ell''}(z) \\ \cos m''\alpha x \sin n'\beta y\, f_{\ell'}(z) \\ \sin m'\alpha x \cos n'\beta y\, f_{\ell''}(z) \\ \sin m''\alpha x \cos n''\beta y\, f_{\ell'}(z) \\ \sin m'\alpha x \sin n''\beta y\, f_{\ell'}(z) \\ \sin m''\alpha x \sin n'\beta y\, f_{\ell''}(z) \end{cases} \tag{3.12}$$

$$\psi \propto \begin{cases} \cos m'\alpha x \cos n''\beta y\ \sin \ell''\pi(z+\frac{1}{2}) \\ \cos m''\alpha x \cos n'\beta y\ \sin \ell'\pi(z+\frac{1}{2}) \\ \cos m'\alpha x \sin n'\beta y\ \sin \ell'\pi(z+\frac{1}{2}) \\ \cos m''\alpha x \sin n''\beta y\ \sin \ell''\pi(z+\frac{1}{2}) \\ \sin m'\alpha x \cos n''\beta y\ \sin \ell'\pi(z+\frac{1}{2}) \\ \sin m''\alpha x \cos n'\beta y\ \sin \ell''\pi(z+\frac{1}{2}) \\ \sin m'\alpha x \cos n'\beta y\ \sin \ell''\pi(z+\frac{1}{2}) \\ \sin m''\alpha x \cos n''\beta y\ \sin \ell'\pi(z+\frac{1}{2}) \end{cases} \tag{3.13}$$

where m', n', ℓ' represent any odd integers whereas m'', n'', ℓ'' represent any even integers. The fact that (3.2), (3.3) and (3.4) preserve the symmetry of the pure hydrodynamic case allows us to use the same set of interacting components (3.12) and (3.13) in the present analysis. It is easily verified that the set, (3.12) and (3.13), is closed in nonlinear interactions: modes outside this set, for example, $\phi \propto \cos m''\alpha x \cos n'\beta y\, f_{\ell'}(z)$, can never be produced through nonlinear interaction among the modes within the set.

4. Numerical results

For numerical purposes the infinite series in (3.6), (3.7) and (3.8) must be truncated so that only those components with $a_{\ell mn}$, $b_{\ell mn}$ and c_k whose subscripts satisfy

$$\ell + |m| + |n| < N_T, \qquad k < N_T' \tag{4.1}$$

N_t, N_t'	M_t upper branch	M_t lower branch	Number of amplitude coefficients
13,11	2.1389078	1.786455	589
14,12	2.1461879	1.609741	734
15,13	2.1426865	1.701963	930
16,14	2.1442994	1.653146	1127
17,15	2.1441258	1.676721	1383
18,16		1.666144	1640

TABLE 1. The momentum transport: $\alpha = 1.6, \beta = 3.0, R = 600, H = 0$.

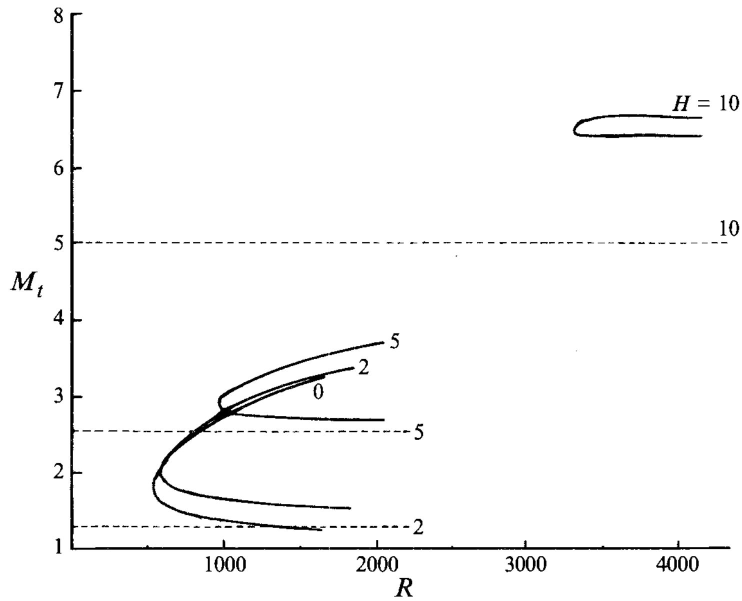

FIGURE 2. The abrupt appearance of the nonlinear solutions: $\alpha = 1.6, \beta = 3.0$ for $H = 0, 2, 5$; $\alpha = 2.0, \beta = 4.5$ for $H = 10$. Straight dashed lines are the momentum transports for undisturbed flows calculated from (2.28).

are taken into account. The resulting finite set of nonlinearly interacting components are solved in terms of their amplitudes, $a_{\ell mn}$, $b_{\ell mn}$ and c_k, numerically by using the Galerkin method combined with the Newton–Raphson method.

Three-dimensional solutions for the purely hydrodynamic case have been presented in Nagata (1990). Since the time of the publication of that paper accuracies have been improved a great deal as shown in table 1. In the table, the momentum transport M_t to be defined by (4.2) is tabulated against (N_T, N_T') for $H = 0$. The relative difference of the momentum transport at successive truncation levels is 0.01% on the upper branch at the truncation level (17,15), whereas it is 0.6% on the lower branch even when the truncation level is increased to (18,16). Although the convergence is slow on the lower branch, the existence of the upper branch assures the existence of the lower branch automatically because of the absence of any bifurcation points on the laminar solution at any R.

In the following calculations the truncation level (14, 12) is employed for economical reasons. The detected solution branches are shown in figure 2 where, as a measure of the nonlinearity the momentum transport at the boundaries,

$$M_t = -\frac{d}{dz}\hat{U}\bigg|_{z=\pm\frac{1}{2}} \tag{4.2}$$

is plotted against the Reynolds number for various Hartmann numbers. The solution branches for $H = 2$ and 5 in figure 2 are obtained from the one for $H = 0$ with $\alpha = 1.6, \beta = 3.0$ (Nagata 1990) by gradually increasing the Hartmann number. It is found that as the Hartmann number is increased the abrupt bifurcation, which can be seen to take place at $R \approx 500$ when $H = 0$, is delayed to a larger Reynolds number indicating the stabilizing effect of the magnetic field. Both the upper and lower branches of the finite-amplitude solutions have more effective momentum transport than the undisturbed laminar flow has for fixed Hartmann numbers. The linear theory (Kakutani 1964) indicates no bifurcation points on the laminar solution branch for Hartmann numbers $0 < H < 7.82$. Therefore, it can be argued that the solution branches in figure 2 originate from the infinite value of the Reynolds number.

Attempts to continue the solution detected at $R = 2000, H = 5$ to larger H without changing the wavenumbers $\alpha = 1.6$ and $\beta = 3.0$ failed. It turned out that for $H = 0$ the closed region for the existence of the solutions in the (α, β)-plane expanded in all directions with a shift of its centre to larger β (see figure 3*a,b*) as the Reynolds number was increased. The corresponding region for $R = 2000, H = 5$ is indicated in figure 3(*c*). When either the Reynolds number or the Hartmann number is increased slightly from $R = 2000, H = 5$, the point (1.6, 3.0) in the (α, β)-plane moves out of the region of existence of the solutions. After learning this fact the solution branch for $H = 10$ in figure 2 was obtained by first gradually increasing the wavenumbers of the soluitons at $H = 5$ in figure 2 to larger values and then increasing the Hartmann number while carefully controlling the Reynolds numbers.

In figure 4 the modification of the mean flows at $R = 1000$ is compared for three different Hartmann numbers $H = 0$, 2 and 5. The larger the magnetic fields become, the weaker are the modifications to the mean flows for the solutions on both upper and lower branches (the distance between the solid curve and the dashed curve in figure 4 gets narrower as the Hartmann number is increased at any position z in the fluid layer). The degree of inflection in the mean velocity profile is less affected by the presence of a stronger magnetic field. As far as the velocity field is concerned, the strength of the modification from the basic laminar flow due to the magnetic field is represented by the change in the Reynolds number measured from the turning point where the abrupt appearance of the nonlinear three-dimensional solutions takes place. The distance is shorter for a stronger magnetic field at a fixed Reynolds number (see figure 2).

As in the case of purely hydrodynamic flows (Nagata 1990), the velocity field is characterized by strong localized currents (not electric currents) in the streamwise direction which meander in a sinusoidal manner in the spanwise direction (see figure 5). Also shown in figure 5 are contours of the helicity Ξ

$$\Xi = \boldsymbol{u} \cdot \nabla \times \boldsymbol{u}. \tag{4.3}$$

The regions of large helicity coincide with those of strong currents. Along a current the helicity changes its sign, taking maxmum and minimum values, indicating a helical fluid motion within the current where the direction of the rotation changes alternatingly along the streamwise direction.

Associated with the helical motion is the generation of the spanwise component of the magnetic field. The magnetic field $\boldsymbol{b}$ in (2.36) is projected on the (y, z)-plane in

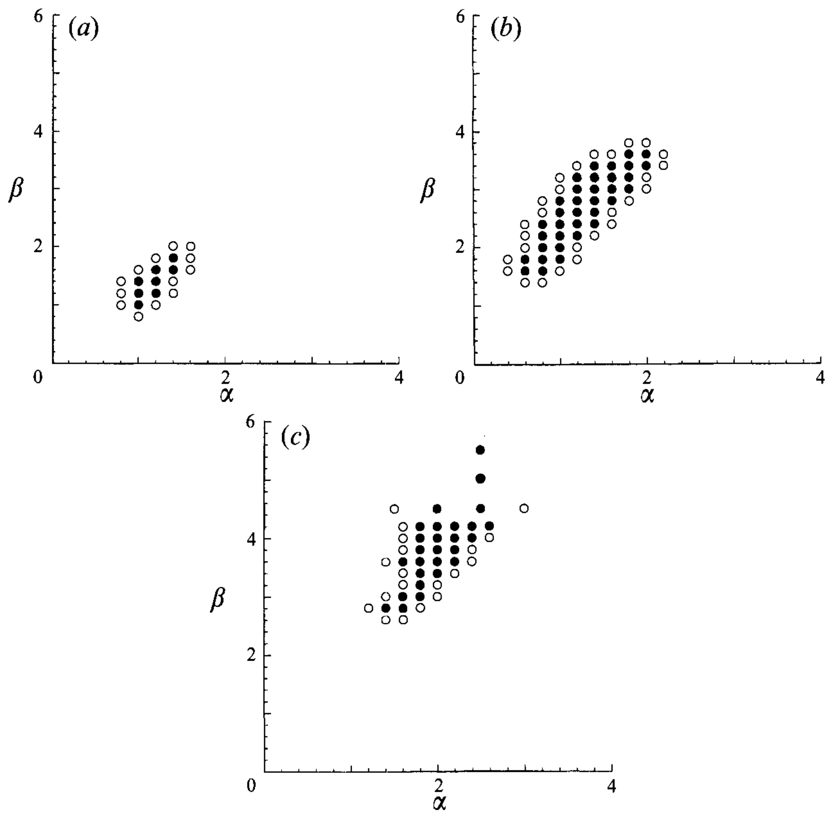

FIGURE 3. The region of existence of the nonlinear solutions. White/black circles indicate whether the nonlinear solutions are/are not detected. (*a*) $R = 525, H = 0$, (*b*) $R = 600, H = 0$, (*c*) $R = 2000, H = 5$.

figure 6. We have solved (2.32) for h in a similar manner as g was obtained in the Appendix. Since h must match the counterpart of the potential magnetic field outside the boundaries, we have used

$$d_z r_{m,n} = \mp \left[(m\alpha)^2 + (n\beta)^2\right]^{1/2} r_{m,n} \tag{4.4}$$

at $z = \pm\frac{1}{2}$, where

$$h = \sum_{m,n} r_{m,n}(z) \exp \mathrm{i}(m\alpha x + n\beta y), \tag{4.5}$$

as an appropriate boundary condition for h. The streamwise vorticity

$$\omega = \hat{\boldsymbol{i}} \cdot \nabla \times \boldsymbol{u} \tag{4.6}$$

is also plotted in figure 6. The whole fluid layer is separated into two distinct sub-layers which have their spanwise magnetic field in opposite directions. With each sub-layer the streamwise vorticity has the same sign.

5. Conclusion

Steady three-dimensional finite-amplitude solutions in plane Couette flows modified by various strengths of transverse magnetic field are presented by a numerical analysis

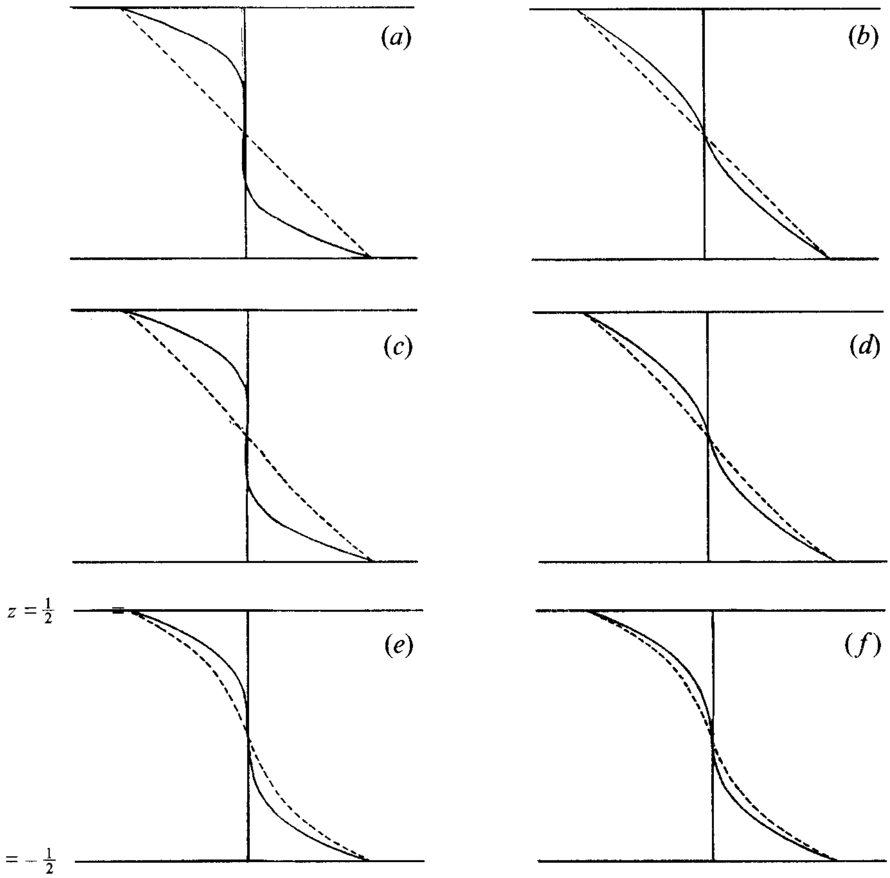

FIGURE 4. The modification of the mean flows. Solid curves correspond to the mean velocity profiles for three-dimensional flows whereas dashed curves correspond to the undisturbed flows (2.28). $\alpha = 1.6, \beta = 3.0, R = 1000$. (*a*) Upper branch, $H = 0$, (*b*) lower branch, $H = 0$, (*c*) upper branch, $H = 2$, (*d*) lower branch, $H = 2$, (*e*) upper branch, $H = 5$, (*f*): lower branch, $H = 5$.

based on the spectral method. The numerical results strongly suggest that nonlinear solutions can exist even if the linear stability analysis predicts no finite values for the critical Reynolds number. Nonlinear solutions appear abruptly at Reynolds numbers of the order of thousands for the range of the Hartmann numbers considered ($0 \leqslant H \leqslant 10$). The minimum Reynolds number, R_{min}, for the existence of the steady three-dimensional solutions seems to increase monotonically from about 500 (at $H = 0$) as H is increased. Therefore, the magnetic field has a stabilizing effect on three-dimensional finite-amplitude flows for all $H(\leqslant 10)$ in contrast to 'a rare example of the destabilizing effect of the magnetic field on parallel flows of an electrically conducting fluid', described by the linear theory (Kakutani 1964). The critical Reynolds number, R_c, predicted by the linear theory is incredibly large: $R_c = 1.52 \times 10^6$ when $H = 10.8$. In our nonlinear analysis $R_{min} \approx 3300$ at $H = 10$. The practicality of the linear theory is doubtful.

The stability analysis of the hydromagnetic solutions obtained is outside the scope of the present investigation (one could argue naively that the upper branch bifurcating from infinity is stable, whereas the lower branch is unstable). In the hydrodynamic cases of Taylor–Couette flow and rotating plane Couette flow, attempts have been made to calculate the growth rate of the general three-dimensional perturbations

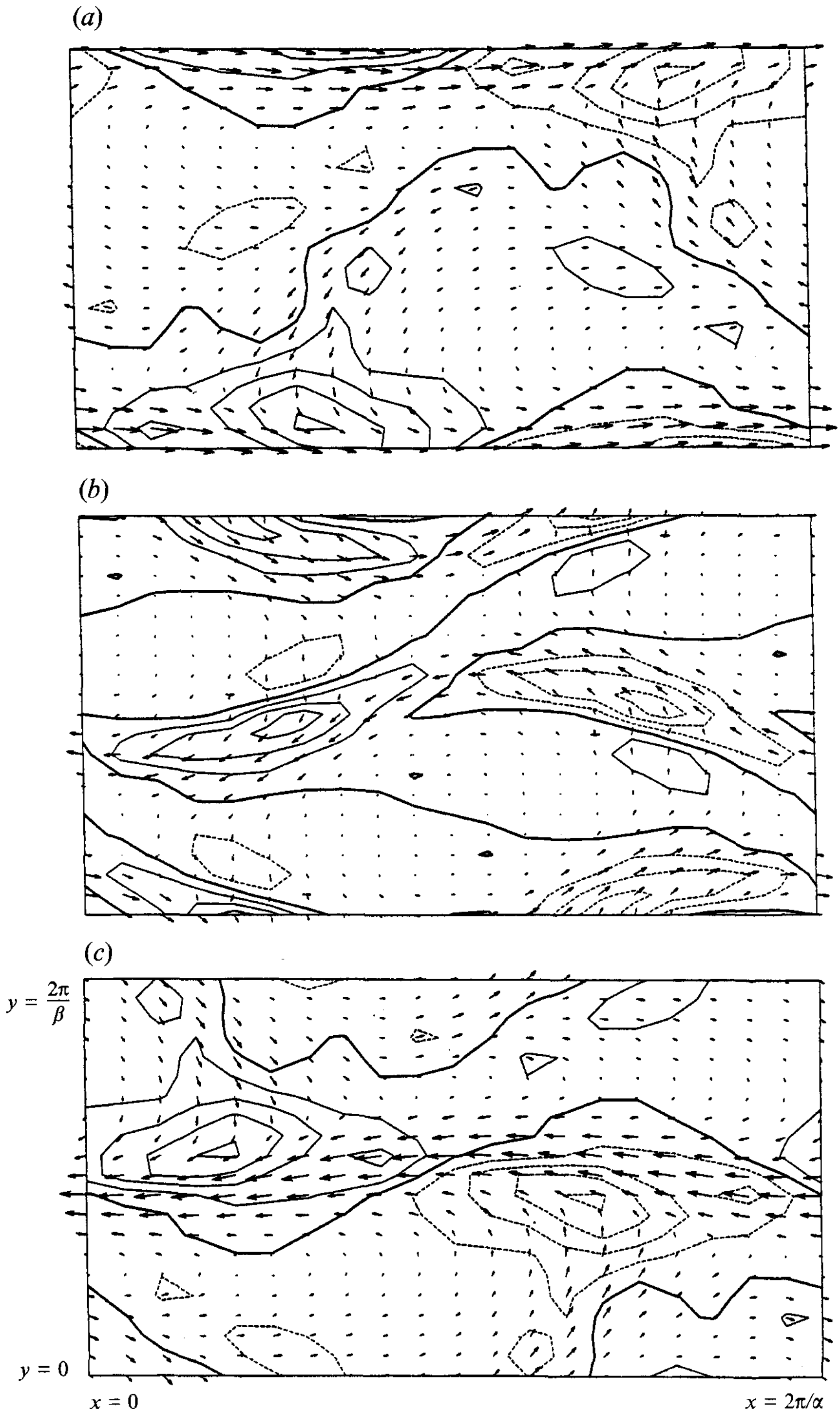

FIGURE 5. The projection of the steady velocity field indicated by arrows on the planes z=const, overlaid with the contours of the helicity Ξ. Solid thick curves correspond to $\Xi = 0$. Positive/negative is helicity indicated by thin solid/dashed curves with the increment $\Delta\Xi = 10^5$. $\alpha = 1.6, \beta = 3.0, R = 2000, H = 5$. (a) $z = -0.25$, (b) $z = 0$, (c) $z = 0.25$.

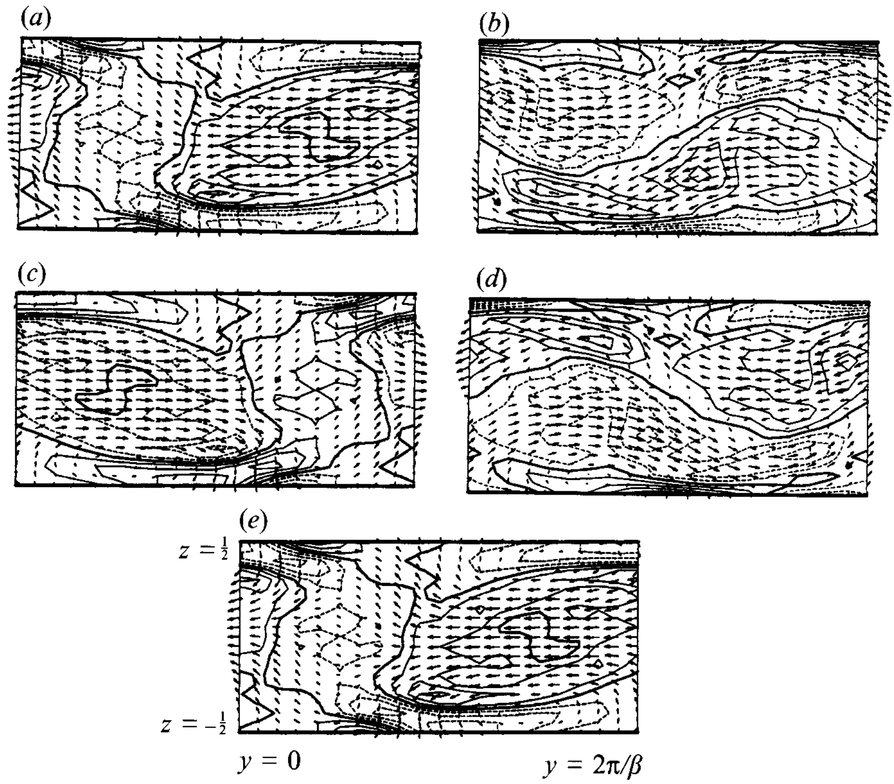

FIGURE 6. The spanwise magnetic field and the streamwise vorticity ω. The projection of the perturbed magnetic field on the (y, z)-plane is indicated by the arrows. Solid thick curves correspond to $\omega = 0$. Positive/negative ω is indicated by thin solid/dashed curves with the increment $\Delta\omega = 400$. $\alpha = 1.6, \beta = 3.0, R = 2000, H = 5$. (*a*) $x = 0$, (*b*) $x = \pi/2\alpha$, (*c*) $x = \pi/\alpha$, (*d*) $x = 3\pi/2\alpha$, (*e*) $x = 2\pi/\alpha$.

superimposed on the steady three-dimensional solutions for the case of zero rotation rate (Nagata 1993,1995), although the results are as yet inconclusive owing to the lack of accuracy. However, it is worthwhile mentioning that recently Dr A. Lundbladh (private communication through Dr W. Koch at DLR, Germany and Professor D. S. Henningson at FFA, Sweden) has observed stable steady three-dimensional flows at relatively low Reynolds numbers in his direct numerical simulations for hydrodynamic plane Couette flows. They seem to correspond to the solutions discovered by Nagata (1990). Therefore, it is plausible that the steady three-dimensional flows obtained here are also stable and thus physically realizable at least when the Hartmann number is not very large. The steady three-dimensional solutions may play an important role in the transition to turbulence in hydromagnetic plane Couette flows.

Appendix

From (2.33) g is written as

$$g = -\partial_z \nabla^{-2} \psi \tag{A 1}$$

by using the inverse Laplacian ∇^{-2}. Let g be represented by

$$g = \sum_{m,n} q_{m,n}(z) \exp \mathrm{i}(m\alpha x + n\beta y). \tag{A 2}$$

Using (3.7) we rewrite (2.33):

$$d^2_{zz}q_{m,n} - ((m\alpha)^2 + (n\beta)^2)q_{m,n} = -\sum_{\ell} \ell\pi b_{\ell mn} \cos \ell\pi(z + \tfrac{1}{2}), \tag{A 3}$$

which can be solved with the boundary condition

$$q_{m,n}(\pm\tfrac{1}{2}) = 0. \tag{A 4}$$

The solution is given by

$$q_{m,n} = A_{m,n} \sinh(\left[(m\alpha)^2 + (n\beta)^2\right]^{1/2} z) + B_{m,n} \cosh\left(\left[(m\alpha)^2 + (n\beta)^2\right]^{1/2} z\right) + \sum_{\ell} \frac{\ell\pi b_{\ell mn}}{(\ell\pi)^2 + (m\alpha)^2 + (n\beta)^2} \cos \ell\pi(z + \tfrac{1}{2}), \tag{A 5}$$

where

$$A_{m,n} = \text{cosech} \frac{\left[(m\alpha)^2 + (n\beta)^2\right]^{1/2}}{2} \sum_{\ell'} \frac{\ell'\pi b_{\ell' mn}}{(\ell'\pi)^2 + (m\alpha)^2 + (n\beta)^2} \tag{A 6}$$

and

$$B_{m,n} = -\text{sech} \frac{\left[(m\alpha)^2 + (n\beta)^2\right]^{1/2}}{2} \sum_{\ell''} \frac{\ell''\pi b_{\ell'' mn}}{(\ell''\pi)^2 + (m\alpha)^2 + (n\beta)^2} \tag{A 7}$$

with ℓ' and ℓ'' denoting odd and even integers, respectively.

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